

Tunneling of the 3rd Kind: A Test of the Effective Non-locality of Quantum Field Theory

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Integrating out virtual quantum fluctuations in an originally local quantum field theory results in an effective theory which is non-local. In this Letter we argue that tunneling of the 3rd kind — where particles traverse a barrier by splitting into a pair of virtual particles which recombine only after a finite distance — provides a direct test of this non-locality. We sketch a quantum-optical setup to test this effect, and investigate observable effects in a simple toy model.

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The existence of virtual particles — field configurations which do not fulfil the classical equations of motion — is an inherent feature of quantum field theory. Such particles lead to a host of interesting and novel effects which find a natural description in the language of effective field theories. One of the earliest quantum field theory calculations that included the effect of virtual particles was the derivation of the Euler-Heisenberg Lagrangian [1–3]. The theory included higher dimensional operators to characterize the effects of virtual fermion loops in light propagation, allowing photons an effective self-interaction [see Fig. 1(a)]. The Uehling potential [4] is a correction to the electrostatic potential caused by an electron loop similar to that shown in Fig. 1(a) (but without the background field) and may be measured as a contribution to the Lamb shift of the hydrogen spectrum, itself an effect of virtual particles. Another example arises in calculating the van der Waals force between two polarizable molecules. Taking into account retardation effects in the calculation yields the Casimir-Polder force [5, 6] [see Fig. 1(b)]. In each of these examples the effects of virtual particles are manifest either in the form of *local* higher dimensional operators (Euler-Heisenberg), or in generating contributions to effective potentials between two particles (Uehling potential and Casimir-Polder force). In this Letter we describe an effect directly demonstrating that the presence of loops also leads to an effective *non-locality* in the propagation of particles and discuss how this effect could be realized in low energy experimental configurations.

The starting point of most quantum field theories is a classical *local* field theory. The action $S \equiv \int d^4x \mathcal{L}$ arises as an integral over a local Lagrangian in which all fields are evaluated at the same point, and typically no more than two derivatives appear. As an example, we consider an interaction of three scalar fields $\phi_{a,b,c}$ (to connect with what follows we take $\phi_{a,b}$ complex and ϕ_c real),

$$S_{\text{int}} = - \int d^4x \frac{\lambda}{\hbar^2 c^2} \phi_a(x) \phi_b^*(x) \phi_c(x) + \text{c.c.} \quad (1)$$

Together with standard kinetic terms and possible couplings

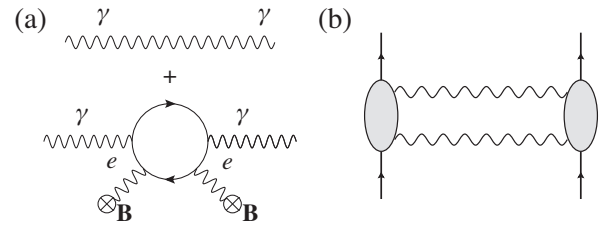


FIG. 1. (a) Contributions to the Euler-Heisenberg Lagrangian. The first term is the classical piece and the second corresponds to the effect of virtual particles in an external magnetic field \mathbf{B} . (b) Leading order contribution to the Casimir-Polder force between two polarizable molecules. The interaction of the photon with the molecule is indicated by the grey blob.

to external potentials the action described by Eq. (1) is entirely local. Consequently the equation of motion for a field at a given space-time point contains only the field and its derivatives at that point. The future evolution is entirely determined by the field value at this point and its immediate surroundings. Alternatively, upon second quantization this becomes a theory of point-like particles with contact interactions. Quantum fluctuations allow for the production of virtual particles that exist briefly and then vanish. For example, with (1) a particle of type ϕ_a can split into a transient ϕ_b and ϕ_c which subsequently recombine into the original ϕ_a , as shown in Fig. 2(a). Although originating entirely from local interactions the effect on the propagation of the *real* particle ϕ_a is *non-local*; it seems to disappear at some point only to reappear elsewhere.

A convenient and illuminating way to account for virtual particle effects is to use an appropriate effective action. The effects of virtual fluctuations are encoded in modified propagation and interaction terms (for simplicity we consider the theory in 1 + 1 dimensions and in the non-relativistic limit)

$$S_{\text{eff}} = \iiint dt dt' dx dx' \phi_a^*(x', t') \left\{ \delta(x - x') \delta(t - t') \times \left[i\hbar \partial_t + \frac{\hbar^2}{2m} \partial_x^2 - V_a(x) \right] - \Pi(x, x', t - t') \right\} \phi_a(x, t) \quad (2)$$

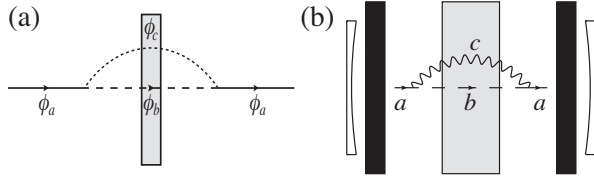


FIG. 2. Schematic processes for tunneling of the 3rd kind. (a) The incoming real particle ϕ_a splits into two virtual particles ϕ_b and ϕ_c which do not interact with the barrier (grey shaded). After traversing the barrier they recombine into ϕ_a . In this way the particle ϕ_a can “tunnel” through the barrier. (b) The incoming atom a emits a virtual photon c and goes to an excited state b . After traversing the barrier they recombine and return to the original state. The mirrors on both side represent an optical cavity used to enhance the coupling of photons to the atom. The black walls denote potential barriers blocking both a and b , confining the atom and preventing contact with the mirrors.

plus terms involving ϕ_b and ϕ_c that we ignore in the following. The terms $\propto \delta(x - x')\delta(t - t')$ are the non-relativistic local kinetic and potential terms. The next term accounts for the effects of virtual particles [see, e.g., Fig. 2(a)]. In particular when $\Pi(x, x', t - t') \not\propto \delta(x - x')$ this describes a spatial non-locality in the effective propagation of ϕ_a . Assuming time-independent potentials $\Pi(x, x', t - t')$ depends only upon the time difference. In this Letter we present a direct way to demonstrate this effective non-locality, as sketched in Fig. 2(a). We start with a particle of type ϕ_a , which encounters a potential barrier (grey). In ordinary quantum mechanics, if the barrier is of finite height and width there is a non-zero tunneling probability; if the barrier is infinitely high this probability is zero. In quantum field theory (QFT), however, the particle may also split into a pair of virtual particles ϕ_b and ϕ_c [7]. If these particles interact only very weakly with the barrier they can traverse the barrier and then recombine into a real ϕ_a particle; the real particle ϕ_a appears to have vanished on one side of the barrier only to reemerge on the other — a clearly non-local effect which we call “tunneling of the 3rd kind” (T3K). We propose a toy-model setup with atoms within an electromagnetic resonator that realizes the essential ingredients necessary for T3K, thereby allowing for a direct test of the effective non-locality of quantum field theory. We then study T3K in this toy-model and calculate the transition probability.

In the Standard Model the only particles suitable as intermediate states for T3K are neutrinos; the probability, however, is prohibitively small [7]. A search for T3K can in fact be sensitive to as yet undiscovered beyond Standard Model particles. Such searches may even improve on bounds from cosmology [8]. Fortunately, it is not necessary to use elementary particles to test T3K and therefore the corresponding non-locality; we can instead use bound states such as atoms or molecules, which can realize all necessary features. Consider specifically the process depicted in Fig. 2(b): an atom emits a virtual photon and goes into a (virtual) excited state; the photon is subsequently reabsorbed and the atom returns to

its original state. This is a loop process closely analogous to that depicted in Fig. 2(a) and represents a quantum field theory correction to the propagation of the atom, which is exactly the type of process we expect for T3K and for generating the effective non-locality of QFT. The only missing ingredient is for the excited atom and the virtual photon to be able to penetrate the barrier, but the atom in its original state being unable to do so. This may be realized within cold-atom systems, where atoms are commonly trapped by potentials generated by electromagnetic fields [9–11]; such potentials may be used to produce a barrier. Indeed such systems of barriers have been used, e.g. in [12], to demonstrate ordinary tunneling of atoms. Potentials of this type can be highly sensitive to the internal state of the atom; accordingly, one may arrange for a situation such that the ground-state atom is blocked but a virtual photon and a suitable excited state is not. Finally, we consider two further refinements of the setup that simplify the calculation and may also aid practical realization. Firstly, we place the whole system within an optical or microwave cavity that allows only for one relevant resonance mode [13]. This can also increase the coupling to the atom thereby making the effect larger. Secondly, as the atoms (both ground and excited states) should preferably not hit the cavity mirrors, we include two further potential barriers to confine the atoms within the cavity. We therefore have two potential wells separated by a central barrier, which is, however, transparent to the excited state [see Fig. 2(b)]. The question of T3K can now be rephrased as: if we start at time $t = 0$ with an atom in the ground state, entirely localized on the left hand side, what is the probability to find it at a later time $t > 0$ in the right hand side potential well?

Let us first link the tunneling rate to the non-local term in our effective Lagrangian (2). We use an eigenbasis expansion of the field ϕ_a to make explicit the separation between time and position dependence, expanding $\phi_a(x, t) = \sum_n c_n^{(a)}(t)\varphi_n^{(a)}(x)$ in energy eigenstates $\varphi_n^{(a)}(x)$ of the unperturbed system, i.e.,

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V_a(x) \right] \varphi_n^{(a)}(x) = E_n^{(a)}\varphi_n^{(a)}(x). \quad (3)$$

Inserting the eigenbasis expansion into (2), and performing the spatial integrals we complete the transformation between position and energy space, leaving us with

$$S_{\text{eff}} = \sum_n \int dt \left[c_n^{(a)*}(t) \left(i\hbar \frac{d}{dt} - E_n^{(a)} \right) c_n^{(a)}(t) \right] - \sum_{n,n'} \iint dt dt' \left[c_n^{(a)*}(t) \Pi_{n,n'} \delta(t - t') c_{n'}^{(a)}(t') \right], \quad (4)$$

where $\Pi_{n,n'} \equiv \iint dx dx' \varphi_n^{(a)*}(x) \Pi(x, x') \varphi_{n'}^{(a)}(x')$. We have also made a simplifying assumption, appropriate for our system. If the intermediate (virtual) two-particle states have energies very different from the initial and final one-particle states, the former are much shorter-lived, and $\Pi_{n,n'}(t - t') \approx \Pi_{n,n'}\delta(t - t')$. This approximation requires us to stay sufficiently far from

resonances. We determine the equations of motion by taking a functional derivative of (4) with respect to $c_n^{(a)*}(t)$:

$$0 = \frac{\delta S_{\text{eff}}}{\delta c_n^{(a)*}(t)} = \left(i\hbar \frac{d}{dt} - E_n^{(a)} \right) c_n^{(a)}(t) - \sum_{n'} \Pi_{n,n'} c_{n'}^{(a)}(t). \quad (5)$$

For simplicity we describe ϕ_a using only the two lowest energy eigenfunctions of (3); if we further assume V_a to be symmetric about the central barrier between the two potential wells, the ground and first-excited state energy eigenfunctions are symmetric and antisymmetric, respectively. Referring to the associated amplitudes and eigenenergies as $c_{S,A}$ and $E_{S,A}$, the equations of motion reduce to a 2×2 matrix equation:

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_S(t) \\ c_A(t) \end{pmatrix} = \left[\begin{pmatrix} E_S & 0 \\ 0 & E_A \end{pmatrix} + \begin{pmatrix} \Pi_{SS} & 0 \\ 0 & \Pi_{AA} \end{pmatrix} \right] \begin{pmatrix} c_S(t) \\ c_A(t) \end{pmatrix}. \quad (6)$$

Parity symmetry requires $\Pi_{AS} = \Pi_{SA} = 0$, and letting the barrier between the two wells tend to infinity we may set $E \equiv E_S = E_A$; Π_{SS} and Π_{AA} then describe energy shifts caused by the creation and annihilation of virtual particles. As is well known, the energy shift $\delta E = \Pi_{AA} - \Pi_{SS}$ determines the leading order tunneling rate between the two wells, which in our setup occurs solely non-locally via virtual intermediate states of the fluctuating fields. Defining “left” and “right” amplitudes through $c_L \equiv (c_S + c_A)/\sqrt{2}$, $c_R \equiv (c_S - c_A)/\sqrt{2}$, and solving for the initial condition $c_L(0) = 1$, $c_R(0) = 0$, we find $c_R(t) = [e^{-i(E+\Pi_{SS})t/\hbar} - e^{-i(E+\Pi_{AA})t/\hbar}]/2$, and for the tunneling probability

$$P_{\text{T3K}}(t) = |c_R(t)|^2 = \sin^2[(\Pi_{AA} - \Pi_{SS})t/2\hbar]. \quad (7)$$

This is of course exactly the field-theory analogue of Rabi oscillations with the characteristic tunneling rate $\sim \delta E/\hbar$.

Let us now consider a Hamiltonian description, mapping the field-theory formulation onto a toy model with (countably infinite) degrees of freedom. We associate the field ϕ_b with atoms of type b having internal energy $\hbar\Omega_b$. These are subject to the exterior trapping potentials [black walls in Fig. 2(b)] which we model by idealized potential walls $V_b(x)$ at a distance $2\ell + d$,

$$V_b(x) = \begin{cases} 0 & \text{for } -\ell - d/2 \leq x \leq \ell + d/2, \\ \infty & \text{for } |x| > \ell + d/2. \end{cases} \quad (8)$$

Similarly, the field ϕ_a is associated with atoms of type a with internal energy $\hbar\Omega_a$. In addition to the exterior potential walls, the atoms a are subject to the central barrier [grey region in Fig. 2(b)] of thickness d , modelled by

$$U(x) = \begin{cases} \infty & \text{for } -d/2 < x < d/2, \\ 0 & \text{for } |x| \geq d/2. \end{cases} \quad (9)$$

Hence, the total potential experienced by atoms in state a is $V_a(x) = V_b(x) + U(x)$. The corresponding bare eigenenergies for atoms in internal states a, b are therefore given by $\hbar\omega_j^{(a)}$, $\hbar\omega_j^{(b)}$, where

$$\omega_j^{(a)} = \Omega_a + \frac{\hbar\pi^2 j^2}{2m\ell^2}, \quad \omega_j^{(b)} = \Omega_b + \frac{\hbar\pi^2 j^2}{2m(2\ell + d)^2}, \quad (10)$$

and the $\omega_j^{(a)}$ are all doubly degenerate (corresponding to energy levels in the left and right infinite potential wells). For simplicity, we again consider only the two (lowest energy) $j = 1$ spatial modes for the a atoms, which we call $\varphi_L^{(a)}(x)$, $\varphi_R^{(a)}(x)$, also setting $\omega_a \equiv \omega_1^{(a)}$; we call the spatial modes for the b atoms $\varphi_j^{(b)}(x)$. We further assume that only a single relevant cavity mode couples the a and b atoms, contributing $\hbar\omega_c \hat{c}^\dagger \hat{c}$ to the system Hamiltonian, where \hat{c}^\dagger creates a photon in the cavity mode function $C(x)$, and ω_c is the corresponding frequency. We consider a standard dipole coupling between the atoms and the cavity photons, and, formulated as a non-relativistic quantum field theory, the complete model Hamiltonian is then

$$\hat{H} = \sum_{\sigma=L,R} \hbar\omega_a \hat{a}_\sigma^\dagger \hat{a}_\sigma + \sum_{j=1}^{\infty} \hbar\omega_j^{(b)} \hat{b}_j^\dagger \hat{b}_j + \hbar\omega_c \hat{c}^\dagger \hat{c} + \sum_{\sigma=L,R} \sum_{j=1}^{\infty} \hbar g_{\sigma j} (\hat{c} + \hat{c}^\dagger) (\hat{a}_\sigma^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{a}_\sigma), \quad (11)$$

where \hat{a}_σ^\dagger creates an atom in internal state a and spatial mode $\varphi_\sigma^{(a)}(x)$, and \hat{b}_j^\dagger creates an atom in internal state b and spatial mode $\varphi_j^{(b)}(x)$, and the couplings are given by the (chosen real) integral

$$g_{\sigma j} = g \int dx \varphi_\sigma^{(a)}(x) \varphi_j^{(b)}(x) C(x), \quad (12)$$

together with a coupling parameter g . Note that we have not carried out any rotating wave approximation (RWA), whereby terms $\propto \hat{c} \hat{a}_\sigma^\dagger \hat{b}_j$, $\hat{c}^\dagger \hat{b}_j^\dagger \hat{a}_\sigma$ would be discarded. The effects of most interest to us in fact manifestly arise from these beyond-RWA terms.

We will consider just a single atom, which may be in either of the two internal states a, b . In this case we may restrict ourselves to a basis $\{|a, \sigma, n\rangle, |b, j, n\rangle\}$, where $\sigma \in \{L, R\}$ and $j \in \{1, 2, 3, \dots\}$ denote the spatial mode, and $n \in \{0, 1, 2, \dots\}$ the number of cavity photons. We can therefore conclude that T3K has taken place if an initially pure $|a, L, 0\rangle$ state is observed in an $|a, R, 0\rangle$ state; to avoid real transitions we consider the situation where there are no photons in the initial and final states. The probability for tunneling of the 3rd kind is then

$$P_{\text{T3K}}(t) = |\langle a, R, 0 | \exp(-i\hat{H}t/\hbar) | a, L, 0 \rangle|^2. \quad (13)$$

An illustrative, although over-naive first approximation for P_{T3K} may be obtained using a basis containing states in only the lowest spatial modes and with at most one photon. The subspace $\{|a, L, 0\rangle, |b, 1, 1\rangle, |a, R, 0\rangle\}$ decouples and we may represent the Hamiltonian in this basis by

$$\hat{H} = \hbar \begin{pmatrix} \omega_a & \tilde{g} & 0 \\ \tilde{g} & \omega_b + \omega_c & \tilde{g} \\ 0 & \tilde{g} & \omega_a \end{pmatrix}, \quad (14)$$

where $\omega_b \equiv \omega_1^{(b)}$ and $\omega_a \equiv \omega_1^{(a)}$ are given in Eq. (10), and $\tilde{g} = g_{L1} = g_{R1}$ is defined in Eq. (12). It is straightforward to solve the time evolution in this case; to lowest order in \tilde{g} :

$$P_{\text{T3K}}(t) = |\sin(\tilde{g}^2 t/\delta) + 2(\tilde{g}/\delta)^2 e^{-i\delta t/2} \sin(\delta t/2)|^2, \quad (15)$$

where $\delta = \omega_b + \omega_c - \omega_a$. Assuming $|\tilde{g}| \ll |\delta|$, there is a slow oscillation with frequency $2\tilde{g}^2/\delta$, and a rapid oscillation with frequency δ , the amplitude of which is suppressed by $(\tilde{g}/\delta)^2$. For small t both parts contribute to a comparable degree, but for large t the first term in (15) dominates. We also see that neglecting the second term [14] in (15) yields the same time-dependence as the result (7) from our Lagrangian formulation, if we identify $\Pi_{AA} - \Pi_{SS} = 2\tilde{g}^2/\delta$. Indeed one may readily verify that $2\tilde{g}^2/\delta$ corresponds to the energy difference between the symmetric and antisymmetric states calculated in second order perturbation theory. Generally, however, the energy required to transfer from $|a, \sigma, 0\rangle$ to $|b, j, 1\rangle$ ($\sim \hbar\delta$) is much larger than the energy splittings between the spatial modes of b atoms $[= \pi^2\hbar^2/2m(2\ell + d)^2 - \text{see (10)}]$. Realistically, we should therefore also include all such spatial modes where $j > 1$. We also note that, if the intermediate state $|b, j, 1\rangle$ has a higher energy than the original state $|a, L, 0\rangle$, i.e., $\delta > 0$, then one would expect from the Heisenberg uncertainty principle that the tunneling rate is exponentially suppressed with increasing energy difference and with increasing wall thickness. This is not exhibited by (15).

Including higher order spatial modes of the virtual b state causes the matrix (14) to become infinite dimensional, but we can nevertheless obtain a result in second order perturbation theory. We consider a simplified situation where the cavity mode $C(x)$ is sufficiently slowly varying for us to take $\tilde{g}_0 \equiv gC(x)$ in (12) to be constant. Second order perturbation theory then yields, for $\delta E = \Pi_{AA} - \Pi_{SS}$,

$$\delta E = \tilde{g}_0^2 \frac{8\ell m \sinh^2(\ell \sqrt{2m\Delta/\hbar}) \sinh^{-1}([2\ell + d] \sqrt{2m\Delta/\hbar})}{\hbar^{3/2}\pi^2 \sqrt{2m\Delta}(1 + 2\ell^2 m\Delta/\hbar\pi^2)^2}, \quad (16)$$

where $\Delta = \Omega_b + \omega_c - \omega_a$.

For $\Delta > 0$ and in the limit $\ell, d \gg \sqrt{\hbar/m\Delta}$, (16) reduces to

$$\delta E \approx \tilde{g}_0^2 \frac{4\ell m \exp(-d \sqrt{2m\Delta/\hbar})}{\hbar^{3/2}\pi^2 \sqrt{2m\Delta}(1 + 2\ell^2 m\Delta/\hbar\pi^2)^2}, \quad (17)$$

which exhibits the expected exponential behavior in the wall thickness. The exponential mass dependence makes an experiment challenging. For example, a single Caesium atom has a mass $\sim 130 \text{ GeV}/c^2$. To obtain a reasonable tunneling rate for a transition frequency of 350 THz ($F = 4 \rightarrow 5$ atomic transition) would require barrier widths $d \lesssim 10 \text{ pm}$ — certainly not straightforwardly realizable with optical lasers.

Experimentally, the situation where the energy of internal state a is higher than that of state b with an additional photon is more promising. We can determine δE perturbatively in the same way as above, which amounts to replacing all occurrences of Δ in (16) with $-|\Delta|$. In this way we obtain

$$\delta E = \tilde{g}_0^2 \frac{4\ell m \sin^2(\ell \sqrt{2m|\Delta|/\hbar}) \sin^{-1}([2\ell + d] \sqrt{2m|\Delta|/\hbar})}{\hbar^{3/2}\pi^2 \sqrt{2m|\Delta|}(1 - 2\ell^2 m|\Delta|/\hbar\pi^2)^2}. \quad (18)$$

Note that in this scenario the energy difference no longer decays exponentially with the wall thickness, making the system

potentially realizable. Fundamentally this is the same process as in the $\Delta > 0$ case; although a real decay with subsequent absorption is energetically possible, the cavity only allows for “wrong frequency photons” [15] such that $\Delta \neq 0$. Hence the intermediate state has a different energy from the initial state and is virtual in this sense.

We have examined the realization of tunneling of the third kind in a toy quantum-optical system. In particular, we have described a setup where atoms transit from one side to the other of an (infinite) potential double well via an intermediate virtual 2-particle (atom-photon) state. From the point of view of the original atom this transition is non-local, with the atom “disappearing” on one side and emerging on the other. In this Letter we have considered a situation far away from resonances. The resonant case may exhibit additional interesting features; in particular, the approximate time-locality may not be valid, leading to further interesting effects. Moreover, the resonant case could simulate a highly topical “light-shining-through-walls” experiment [16] used to search for new light particles beyond the Standard Model, such as axions and hidden photons [17].

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